Graph Filter Banks with $M$-Channels, Maximal Decimation, and Perfect Reconstruction

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Outline

1. Introduction to Graph Signal Processing
   - Graph Signals

2. Purpose: $M$-Channel Filter Banks
   - Extending Classical Tools to Graph Case

3. Multirate Processing of Graph Signals
   - Decimation & Expansion
   - $M$-Block Cyclic Graphs
   - Concept of Spectrum Folding
   - More Results for $M$-Block Cyclic

4. Some Examples
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4. Some Examples
What is a Graph Signal?


What is a Graph Signal?

A is the adjacency matrix\(^1,2,3\),

\[
A \in \mathcal{M}^N
\]

\[
x = \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_i \\
    \vdots \\
    x_N
\end{bmatrix} \in \mathbb{C}^N
\]

---


What is a Graph Signal?

A is the adjacency matrix, \( A \in M^N \),

A is considered as the graph operator, with

\[
A = V \Lambda V^{-1}
\]
What is a Graph Signal?

A is the adjacency matrix\(^1,2,3\), \(A \in M^N\)

\(A\) is considered as the graph operator, with \(A = V \Lambda V^{-1}\)

\(V\) = graph Fourier Basis, \(V^{-1}\) = graph Fourier Transform

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Maximally Decimated $M$-Channel Filter Banks

Diagram:

- **Analysis Filter Bank**
  - $x$ \( \rightarrow H_0 \rightarrow D \rightarrow x_0 \rightarrow D^T \rightarrow F_0 \rightarrow y_0 \rightarrow y \)
  - $H_0, D, F_0$
  - $x \rightarrow H_1 \rightarrow D \rightarrow x_1 \rightarrow D^T \rightarrow F_1 \rightarrow y_1 \rightarrow \ldots$
  - $H_{M-1}, D, F_{M-1}$

- **Synthesis Filter Bank**
  - $y \rightarrow F_0 \rightarrow D^T \rightarrow x_{M-1} \rightarrow D \rightarrow H_{M-1} \rightarrow \ldots$
  - $y \rightarrow F_1 \rightarrow D^T \rightarrow x_1 \rightarrow D \rightarrow H_1 \rightarrow \ldots$
  - $y \rightarrow F_{M-1} \rightarrow D^T \rightarrow x_0 \rightarrow F_0 \rightarrow \ldots$

- $x$, $y$, $D$, $D^T$, $F_0$, $F_1$, $F_{M-1}$, $H_0$, $H_1$, $H_{M-1}$

- **Symbols:**
  - $M$-Channel Filter Banks
  - Maximally Decimated
  - $x$, $y$, $D$, $D^T$, $F_0$, $F_1$, $F_{M-1}$, $H_0$, $H_1$, $H_{M-1}$
$M$-Channel Filter Banks (Brute-Force)

![Diagram of $M$-Channel Filter Banks](image)

- **Analysis Filter Bank**
  - $x$ \(\rightarrow\) $x_0$ \(\rightarrow\) $x_1$ \(\rightarrow\) $x_{M-1}$
  - $DH_0$ to $DH_{M-1}$

- **Synthesis Filter Bank**
  - $y_0$ \(\rightarrow\) $y_1$ \(\rightarrow\) $y_{M-1}$
  - $F_0D^T$ to $F_{M-1}D^T$
M-Channel Filter Banks (Brute-Force)

\[
\begin{align*}
H_{\text{anal}} &= \begin{bmatrix}
DH_0 \\
\vdots \\
DH_{M-1}
\end{bmatrix}, & F_{\text{syn}} &= \begin{bmatrix}
F_0 D^T \\
\vdots \\
F_{M-1} D^T
\end{bmatrix}
\end{align*}
\]
$M$-Channel Filter Banks (Brute-Force)

Analysis filter bank

$$H_{anal} = \begin{bmatrix} DH_0 \\ \vdots \\ DH_{M-1} \end{bmatrix}, \quad F_{syn} = \begin{bmatrix} F_0 D^T \\ \vdots \\ F_{M-1} D^T \end{bmatrix}$$

$$F_{syn} H_{anal} = I$$
Problems with Brute-Force

\[ F_{\text{syn}} H_{\text{anal}} = I \]
Problems with Brute-Force

\[ F_{\text{syn}} H_{\text{anal}} = I \]

\[ F_{\text{syn}} = H_{\text{anal}}^{-1} \]
Problems with Brute-Force

\[ F_{\text{syn}} H_{\text{anal}} = I \]

\[ F_{\text{syn}} = H_{\text{anal}}^{-1} \]

Filter Design

\[ \Leftrightarrow \]

Matrix Inversion Problem
Problems with Brute-Force

\[ F_{\text{syn}} \cdot H_{\text{anal}} = I \]

\[ F_{\text{syn}} = H_{\text{anal}}^{-1} \]

Relation with \( A \)?

Filter Design

\( \uparrow \)

Matrix Inversion Problem
Problems with Brute-Force

\[
\begin{align*}
F_{syn} \cdot H_{anal} &= I \\
F_{syn} &= H_{anal}^{-1}
\end{align*}
\]

Filter Design

\[ \uparrow \]

Matrix Inversion Problem

Relation with \( A \)?

Computational Complexity = \( O(N^2) \)
Polynomials

\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]
Polynomials

\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]

\[ x \xrightarrow{N} A \xrightarrow{N} A \xrightarrow{\ldots} A \xrightarrow{D} x_k \]
\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]

Cost: \( LN^2 + LN \)
Polynomials

\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]

Cost: \( LN^2 + LN \)

\( A \) has simple entries, e.g. \( \{0, 1, -1\} \), \( \Rightarrow A x \) has negligible complexity
Polynomials

\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]

Cost: \( LN^2 + LN \)

\( A \) has simple entries, e.g. \( \{0, 1, -1\} \), \( \Rightarrow A x \) has negligible complexity

Cost of \( H_k(A) \) is \( O(LN) \) v.s. \( O(N^2) \) in brute-force
Polynomials

\[ H_k(A) = h_k(0) A^0 + h_k(1) A^1 + h_k(2) A^2 + \ldots + h_k(L) A^L \]

Cost: \( LN^2 + LN \)

\( A \) has simple entries, e.g. \{0, 1, -1\}, \( A x \) has negligible complexity

Cost of \( H_k(A) \) is \( O(LN) \) v.s. \( O(N^2) \) in brute-force

\( L \ll N \)
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$M$-Fold Decimation

$$D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/M}$$
$M$-Fold Decimation

$D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/M}$

$D \mathbf{x}$
$M$-Fold Decimation

$$D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/M}$$

Which samples to keep?
$M$-Fold Decimation

\[ D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/M} \]

\[ D x \]

Which samples to keep?

Assume an appropriate permutation (labeling) of the nodes

Keep the first $N/M$ samples
**M-Fold Decimation**

\[ D : \mathbb{C}^N \rightarrow \mathbb{C}^{N/M} \]

\[ D \mathbf{x} \]

Which samples to keep?

Assume an appropriate permutation (labeling) of the nodes

Keep the first \( \frac{N}{M} \) samples

Why? Labelling of the nodes is arbitrary!
**M-Fold Decimation**

\[ D : C^N \rightarrow C^{N/M} \]

\[ D \mathbf{x} \]

Which samples to keep?

Assume an appropriate permutation (labeling) of the nodes

Keep the first \( N/M \) samples

Why? Labelling of the nodes is arbitrary!

**Definition (Canonical Decimator)**

\[ D = \begin{bmatrix} I_{N/M} & 0_{N/M} & \cdots & 0_{N/M} \end{bmatrix} \in C^{(N/M) \times N}, \]

which retains the first \( N/M \) samples of the given graph signal.
$M$-fold Expansion

$U : C^{N/M} \rightarrow C^N$
$M$-fold Expansion

$$U : C^{N/M} \rightarrow C^{N}$$

Given the decimator $D = [I_{N/M} \ 0_{N/M} \ \cdots \ 0_{N/M}]$

$$U = D^T = \begin{bmatrix}
I_{N/M} \\
0_{N/M} \\
\vdots \\
0_{N/M}
\end{bmatrix} \in \mathbb{C}^{N \times (N/M)}$$
$M$-fold Expansion

$$U : \mathbb{C}^{N/M} \rightarrow \mathbb{C}^N$$

Given the decimator $D = \begin{bmatrix} I_{N/M} & 0_{N/M} & \cdots & 0_{N/M} \end{bmatrix}$

$$U = D^T = \begin{bmatrix} I_{N/M} \\ 0_{N/M} \\ \vdots \\ 0_{N/M} \end{bmatrix} \in \mathbb{C}^{N \times (N/M)}$$

Upsample-then-downsample, $DU$, is unity

$$DU = I$$
$M$-fold Expansion

\[ U : C^{N/M} \rightarrow C^N \]

Given the decimator \( D = \begin{bmatrix} I_{N/M} & 0_{N/M} & \cdots & 0_{N/M} \end{bmatrix} \)

\[ U = D^T = \begin{bmatrix} I_{N/M} \\ 0_{N/M} \\ \vdots \\ 0_{N/M} \end{bmatrix} \in C^{N\times(N/M)} \]

Upsample-then-downsample, \( DU \), is unity

\[ DU = I \]

\[ D^T = \arg \min_U \|U\|_F \quad \text{s.t.} \quad DU = I \]
\( M \)-block cyclic graphs

---


\(^b\)
$M$-block cyclic graphs

$$A = \begin{bmatrix}
0 & \cdots & 0 & A_M \\
A_1 & \cdots & 0 & 0 \\
\vdots & \ddots & 0 & \vdots \\
0 & \cdots & A_{M-1} & 0
\end{bmatrix} \in \mathcal{M}^N$$

$$A_j \in \mathcal{M}^{N/M}$$

(Under suitable permutation)


\footnote{Teke & Vaidyanathan}
$M$-block cyclic graphs

\[ A = \begin{bmatrix}
0 & \cdots & 0 & A_M \\
A_1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & A_{M-1} & 0 \\
\end{bmatrix} \in \mathcal{M}^N \]

\[ A_j \in \mathcal{M}^{N/M} \]

(Under suitable permutation)

\[^{a}O.\ Teke\ and\ P.\ P.\ Vaidyanathan.\ "Extending\ Classical\ Multirate\ Signal\ Processing\ Theory\ to\ Graphs".\ IEEE\ Trans.\ Signal\ Process.\ (Submitted).\]

\[^{b}\]
If a graph is $M$-block cyclic, then it is $M$-partite, but not vice-versa.

$A = \begin{bmatrix}
0 & \cdots & 0 & A_M \\
A_1 & \cdots & 0 & 0 \\
\vdots & \ddots & 0 & \vdots \\
0 & \cdots & A_{M-1} & 0
\end{bmatrix} \in \mathbf{M}^N$

$A_j \in \mathbf{M}^{N/M}$

(Under suitable permutation)

---


$^b$
If a graph is $M$-block cyclic, then it is $M$-partite, but not vice-versa.

A graph is 2-block cyclic if and only if it is bi-partite.

$A = \begin{bmatrix} 0 & \cdots & 0 & A_M \\ A_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{M-1} & 0 \end{bmatrix} \in \mathcal{M}^N$

$A_j \in \mathcal{M}^{N/M}$ (Under suitable permutation)

If a graph is $M$-block cyclic, then it is $M$-partite, but not vice-versa.

A graph is 2-block cyclic if and only if it is bi-partite.

An $M$-block cyclic graph is necessarily a directed graph for $M > 2$.

---

\[ A = \begin{bmatrix}
0 & \cdots & 0 & A_M \\
A_1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & A_{M-1} & 0
\end{bmatrix} \in \mathcal{M}^N \]

\[ A_j \in \mathcal{M}^{N/M} \]

(Under suitable permutation)

---


\[^b\text{ICASSP 2016 13 / 25}\]
**M**-block cyclic graphs

\[
A = \begin{bmatrix}
0 & \ldots & 0 & A_M \\
A_1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & A_{M-1} & 0
\end{bmatrix} \in \mathcal{M}^N
\]

\[
A_j \in \mathcal{M}^{N/M}
\]

(Under suitable permutation)

1. If a graph is **M**-block cyclic, then it is **M**-partite, but not vice-versa.
2. A graph is 2-block cyclic if and only if it is bi-partite.
3. An **M**-block cyclic graph is necessarily a directed graph for **M** > 2.
4. A cyclic graph of size **N**, \( C_N \), is an **M**-block cyclic graph for all **M** that divides **N**.

---

$M$-block cyclic graphs

$A = \begin{bmatrix}
0 & \cdots & 0 & A_M \\
A_1 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & A_{M-1} & 0
\end{bmatrix} \in \mathcal{M}^N$

$A_j \in \mathcal{M}^{N/M}$

(Under suitable permutation)

1. If a graph is $M$-block cyclic, then it is $M$-partite, but not vice-versa.
2. A graph is 2-block cyclic if and only if it is bi-partite.
3. An $M$-block cyclic graph is necessarily a directed graph for $M > 2$.
4. A cyclic graph of size $N$, $C_N$, is an $M$-block cyclic graph for all $M$ that divides $N$.
5. Unique eigenvalue & eigenvector structure $\begin{bmatrix} a \\ b \end{bmatrix}$

---


Spectrum Folding - Aliasing

Let $x$ be a graph signal

$$y = D^T D x$$
Let $x$ be a graph signal

$$y = D^T D x$$

$$x = \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_{\frac{N}{M}} \\
    x_{\frac{N}{M} + 1} \\
    \vdots \\
    \vdots \\
    \vdots \\
    x_N
\end{bmatrix}$$
Let $x$ be a graph signal

$$y = D^T D x$$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{N/M} \\ x_{N/M} + 1 \\ \vdots \\ x_N \end{bmatrix}$

$D \rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_{N/M} \end{bmatrix}$
Spectrum Folding - Aliasing

Let $x$ be a graph signal

$$y = D^T D x$$

$$x = \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_{\frac{N}{M}} \\
  x_{\frac{N}{M}+1} \\
  \vdots \\
  \vdots \\
  \vdots \\
  x_N \\
\end{bmatrix} \xrightarrow{D} \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_{\frac{N}{M}} \\
\end{bmatrix} \xrightarrow{D^T} \begin{bmatrix}
  x_1 \\
  \vdots \\
  x_{\frac{N}{M}} \\
  0 \\
  \vdots \\
  \vdots \\
  0 \\
\end{bmatrix} = y$$
Let $x$ be a graph signal

$$y = D^T D x$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{\frac{N}{M}} \\ x_{\frac{N}{M} + 1} \\ \vdots \\ \vdots \\ \vdots \\ x_N \end{bmatrix} \xrightarrow{D} \begin{bmatrix} x_1 \\ \vdots \\ x_{\frac{N}{M}} \end{bmatrix} \xrightarrow{D^T} \begin{bmatrix} x_1 \\ \vdots \\ x_{\frac{N}{M}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = y$$

What is the relation between $\hat{x}$ and $\hat{y}$?
What is the relation between $\hat{x}$ and $\hat{y}$?

\[
\hat{y} = V^{-1} D^T D V \hat{x}
\]
What is the relation between $\hat{x}$ and $\hat{y}$?

$$\hat{y} = V^{-1} D^T D V \; \hat{x}$$
What is the relation between $\hat{x}$ and $\hat{y}$?

$$\hat{y} = V^{-1} D^T D V \hat{x}$$
What is the relation between $\hat{x}$ and $\hat{y}$?

$$\hat{y} = V^{-1} D^T D V \hat{x}$$
What is the relation between $\hat{x}$ and $\hat{y}$?

$$\hat{y} = V^{-1} D^T D V \hat{x}$$

No "simple" relation in general!
Spectrum Folding on $\mathcal{M}$-Block Cyclic Graphs

\[
\begin{pmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\vdots
\end{pmatrix}

\begin{pmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\vdots \\
\vdots
\end{pmatrix}

1 \overset{\mathcal{M}}{\longrightarrow} 2 \overset{\mathcal{M}}{\overset{\mathcal{M}}{\longrightarrow}} \cdots

Spectrum Folding on $M$-Block Cyclic Graphs

\[
\begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\vdots \\
\end{bmatrix}
\xrightarrow{1/M \sum}
\begin{bmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\vdots \\
\vdots \\
\end{bmatrix}
\]
Spectrum Folding on $\mathcal{M}$-Block Cyclic Graphs

\[
\begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix} \xrightarrow{\frac{1}{M} \sum} \begin{bmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\vdots \\
\vdots \\
\end{bmatrix}
\]
Spectrum Folding on $M$-Block Cyclic Graphs

$$\begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\vdots \\
\vdots \\
\hat{y}_{i,1} = \cdots = \hat{y}_{i,M} = \frac{1}{M} \sum_{j=1}^{M} \hat{x}_{i,j}
\end{bmatrix}$$

$$\begin{bmatrix}
\frac{1}{M} \sum \\
\vdots \\
\frac{1}{M} \sum \\
\vdots \\
\vdots \\
\vdots \\
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M}
\end{bmatrix}$$
Spectrum Folding on $M$-Block Cyclic Graphs

\[
\begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\xrightarrow{\frac{1}{M} \sum}
\begin{bmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\vdots \\
\vdots \\
\vdots
\end{bmatrix}
\]

\[
\hat{y}_{i,1} = \cdots = \hat{y}_{i,M} = \frac{1}{M} \sum_{j=1}^{M} \hat{x}_{i,j}
\]

$M = 2 \leftrightarrow$ Bi-partite

---

Let $\mathbf{x}$ be a graph signal

$$
\hat{x} = \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix}
$$
Let $\mathbf{x}$ be a graph signal

\[
\hat{\mathbf{x}} = \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M}
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M}
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{3,1} \\
\vdots \\
\vdots
\end{bmatrix}
\rightarrow \begin{bmatrix}
\hat{x}_{1,1} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{2,1} \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{3,1} \\
\vdots \\
\vdots
\end{bmatrix}
\]
Let $\mathbf{x}$ be a graph signal

$$\mathbf{x} = \begin{bmatrix} \hat{x}_{1,1} \\ \vdots \\ \hat{x}_{1,M} \\ \hat{x}_{2,1} \\ \vdots \\ \hat{x}_{2,M} \\ \hat{x}_{3,1} \\ \vdots \end{bmatrix} \xrightarrow{k^{th} \text{ band limited}} \begin{bmatrix} \hat{x}_{1,1} \\ 0 \\ 0 \\ \hat{x}_{2,1} \\ 0 \\ 0 \\ \hat{x}_{3,1} \\ \vdots \end{bmatrix} \xrightarrow{D^T D}$$
Let $x$ be a graph signal

\[
\begin{align*}
\hat{x} &= \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix} \\
&\xrightarrow{k^{th} \text{ band limited}} \\
&\begin{bmatrix}
\hat{x}_{1,1} \\
0 \\
0 \\
\hat{x}_{2,1} \\
0 \\
0 \\
\hat{x}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix} \\
&\xrightarrow{D^T D} \\
\hat{y} &= \begin{bmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\hat{y}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix}
\end{align*}
\]
Let $x$ be a graph signal

\[
\hat{x} = \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\hat{x}_{3,1} \\
\vdots
\end{bmatrix}
\xrightarrow{k^{th} \text{ band limited}}
\begin{bmatrix}
\hat{x}_{1,1} \\
0 \\
0 \\
\hat{x}_{2,1} \\
0 \\
0 \\
\hat{x}_{3,1} \\
\vdots
\end{bmatrix}
\xrightarrow{\frac{1}{M} \sum}
\begin{bmatrix}
\hat{y}_{1,1} \\
\vdots \\
\hat{y}_{1,M} \\
\hat{y}_{2,1} \\
\vdots \\
\hat{y}_{2,M} \\
\vdots \\
\hat{y}_{3,1} \\
\vdots
\end{bmatrix}
= \frac{1}{M}
\begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\vdots \\
\hat{x}_{3,1} \\
\vdots
\end{bmatrix}
\]
Interpolation Filter on $M$-Block Cyclic Graphs

\[ \hat{y} = \frac{1}{M} \begin{bmatrix} \hat{x}_{1,1} \\ \vdots \\ \hat{x}_{1,1} \\ \hat{x}_{2,1} \\ \vdots \\ \hat{x}_{2,1} \\ \hat{x}_{3,1} \\ \vdots \\ \vdots \end{bmatrix} \]
Interpolation Filter on $M$-Block Cyclic Graphs

\[
\hat{y} = \frac{1}{M} \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,1} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,1} \\
\hat{x}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix}
\]

\[z = Fy \quad \text{(interpolation)}\]
Interpolation Filter on $M$-Block Cyclic Graphs

$$\hat{y} = \frac{1}{M} \begin{bmatrix} \hat{x}_{1,1} \\ \vdots \\ \hat{x}_{1,1} \\ \hat{x}_{2,1} \\ \vdots \\ \hat{x}_{2,1} \\ \vdots \\ \hat{x}_{3,1} \end{bmatrix} \overset{\text{(interpolation)}}{\rightarrow} \hat{z} = \begin{bmatrix} \hat{x}_{1,1} \\ 0 \\ 0 \\ \hat{x}_{2,1} \\ 0 \\ 0 \\ \hat{x}_{3,1} \\ 0 \\ \vdots \end{bmatrix} = \hat{x}$$
Interpolation Filter on $M$-Block Cyclic Graphs

\[
\hat{y} = \frac{1}{M} \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,1} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{3,1} \\
\vdots \\
\vdots 
\end{bmatrix}
\]

\[\text{z} = Fy\] (interpolation)

\[
\hat{z} = \begin{bmatrix}
\hat{x}_{1,1} \\
0 \\
0 \\
\hat{x}_{2,1} \\
0 \\
0 \\
\hat{x}_{3,1} \\
0 \\
\vdots 
\end{bmatrix} = \hat{x}
\]

Not true for an arbitrary $x$!

For $k^{th}$-band-limited signals, $x$ can be recovered from $Dx$
$M$-Channel Filter-Banks on $M$-Block Cyclic Graphs

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_{1,1} \\ \vdots \\ \hat{x}_{1,M} \\ \hat{x}_{2,1} \\ \vdots \\ \hat{x}_{2,M} \\ \hat{x}_{3,1} \\ \vdots \\ \vdots \end{bmatrix}$$
$M$-Channel Filter-Banks on $M$-Block Cyclic Graphs

\[
\hat{x} = \begin{bmatrix}
\hat{x}_{1,1} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\vdots \\
\cdots
\end{bmatrix}
= \begin{bmatrix}
\hat{x}_{1,1} \\
0 \\
\vdots \\
0 \\
\hat{x}_{2,1} \\
0 \\
\hat{x}_{2,2} \\
\vdots \\
0 \\
\hat{x}_{3,1} \\
0 \\
\hat{x}_{3,2} \\
\vdots \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\vdots \\
0 \\
\hat{x}_{2,M} \\
0 \\
\hat{x}_{2,2} \\
\vdots \\
0 \\
0 \\
0 \\
\hat{x}_{3,M} \\
\vdots
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
0 \\
\vdots \\
0 \\
\hat{x}_{1,M} \\
0 \\
\hat{x}_{1,2} \\
\vdots \\
0 \\
0 \\
\hat{x}_{3,M} \\
\vdots
\end{bmatrix}
\]

$\hat{x}_1 = 1^{st}$ member

$\hat{x}_2 = 2^{nd}$ member

$\hat{x}_M = M^{th}$ member
$M$-Channel Filter-Banks ($k^{th}$-Channel)

$$
\hat{x} = \begin{bmatrix}
\hat{x}_{1,1} \\
\hat{x}_{1,2} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\hat{x}_{2,2} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\hat{x}_{3,2} \\
\vdots \\
\hat{x}_{3,M} \\
\vdots
\end{bmatrix}
$$
$M$-Channel Filter-Banks ($k^{th}$-Channel)

\[
\hat{x} = \begin{bmatrix}
\hat{x}_{1,1} \\
\hat{x}_{1,2} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\hat{x}_{2,2} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\hat{x}_{3,2} \\
\vdots \\
\hat{x}_{3,M} \\
\vdots \\
\end{bmatrix}
\]

\[
H_k \xrightarrow{\hat{x}_k =} \begin{bmatrix}
0 \\
\hat{x}_{1,k} \\
\vdots \\
\hat{x}_{2,k} \\
0 \\
\hat{x}_{3,k} \\
\vdots \\
0 \\
\end{bmatrix}
\]
$M$-Channel Filter-Banks ($k^{th}$-Channel)

\[
\begin{align*}
\hat{x} &= \begin{bmatrix}
\hat{x}_{1,1} \\
\hat{x}_{1,2} \\
\vdots \\
\hat{x}_{1,M} \\
\hat{x}_{2,1} \\
\hat{x}_{2,2} \\
\vdots \\
\hat{x}_{2,M} \\
\hat{x}_{3,1} \\
\hat{x}_{3,2} \\
\vdots \\
\hat{x}_{3,M} \\
\vdots \\
\end{bmatrix} \xrightarrow{H_k} \begin{bmatrix}
0 \\
\hat{x}_{1,k} \\
0 \\
\vdots \\
0 \\
\hat{x}_{2,k} \\
0 \\
\vdots \\
0 \\
\hat{x}_{3,k} \\
0 \\
\vdots \\
0 \\
\end{bmatrix} \xrightarrow{D^T D} \begin{bmatrix}
\hat{x}_{1,k} \\
\vdots \\
\hat{x}_{1,k} \\
\vdots \\
\hat{x}_{1,k} \\
\hat{x}_{2,k} \\
\vdots \\
\hat{x}_{2,k} \\
\vdots \\
\hat{x}_{2,k} \\
\vdots \\
\hat{x}_{3,k} \\
\vdots \\
\end{bmatrix}
\end{align*}
\]
### $M$-Channel Filter-Banks ($k^{th}$-Channel)

\[ \hat{x} = \begin{bmatrix} \hat{x}_{1,1} \\ \hat{x}_{1,2} \\ \vdots \\ \hat{x}_{1,M} \\ \hat{x}_{2,1} \\ \hat{x}_{2,2} \\ \vdots \\ \hat{x}_{2,M} \\ \hat{x}_{3,1} \\ \hat{x}_{3,2} \\ \vdots \\ \hat{x}_{3,M} \\ \vdots \end{bmatrix} \xrightarrow{H_k} \hat{x}_k = \begin{bmatrix} 0 \\ \hat{x}_{1,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{2,k} \\ \vdots \\ 0 \\ 0 \\ \hat{x}_{3,k} \\ \vdots \end{bmatrix} \xrightarrow{D^T D} \frac{1}{M} \hat{y}_k = \begin{bmatrix} \hat{x}_{1,k} \\ \hat{x}_{1,k} \\ \vdots \\ \hat{x}_{2,k} \\ \hat{x}_{2,k} \\ \vdots \\ \hat{x}_{3,k} \\ \hat{x}_{3,k} \end{bmatrix} \xrightarrow{F_k} \hat{z}_k = \begin{bmatrix} 0 \\ \hat{x}_{2,k} \\ \hat{x}_{3,k} \\ \vdots \end{bmatrix} \]
$M$-Channel FB on $M$-Block Cyclic Graphs (Details)

\[ x \xrightarrow{H_0(A)} D \xrightarrow{D^T} F_0(A) \xrightarrow{\text{+}} y \]

\[ H_{M-1}(A) \xrightarrow{D} D^T \xrightarrow{F_{M-1}(A)} \]

\[ H_k \neq \text{needs to have distinct eigenvalues.} \]
$M$-Channel FB on $M$-Block Cyclic Graphs (Details)

$$F_k = M \cdot H_k$$
**$M$-Channel FB on $M$-Block Cyclic Graphs (Details)**

\[ \begin{align*}
  \mathbf{x} & \rightarrow H_0(A) \rightarrow D \rightarrow D^T \rightarrow F_0(A) \rightarrow + \rightarrow \mathbf{y} \\
  & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
  & \quad H_{M-1}(A) \rightarrow D \rightarrow D^T \rightarrow F_{M-1}(A) \\
\end{align*} \]

\[ F_k = M \cdot H_k \]

\[ H_{k-1} = V \left( \mathbf{I} \otimes e_k e_k^T \right) V^{-1} \]
$M$-Channel FB on $M$-Block Cyclic Graphs (Details)

\[ \begin{align*}
x & \xrightarrow{H_0(A)} D \xrightarrow{D^T} F_0(A) + \ldots \\
& \xrightarrow{H_{M-1}(A)} D \xrightarrow{D^T} F_{M-1}(A)
\end{align*} \]

\[ F_k = M H_k \]

\[ H_{k-1} = V \left( I \otimes e_k e_k^T \right) V^{-1} \]

\[ H_k(e^{j\omega}) = \begin{cases} 
1, & \frac{2\pi k}{M} \leq \omega \leq \frac{2\pi (k+1)}{M}, \\
0, & \text{otherwise},
\end{cases} \]
$M$-Channel FB on $M$-Block Cyclic Graphs (Details)

\[ F_k = M \, H_k \]

\[ H_{k-1} = V \left( I \otimes e_k e_k^T \right) V^{-1} \]

\[ H_k(e^{j\omega}) = \begin{cases} 
1, & \frac{2\pi k}{M} \leq \omega \leq \frac{2\pi (k+1)}{M}, \\
0, & \text{otherwise},
\end{cases} \]

\[ H_k = H_k(A) = h_k(0) \, A^0 + h_k(1) \, A^1 + \ldots + h_k(L) \, A^L \]
More Results for $M$-Block Cyclic

$M$-Channel FB on $M$-Block Cyclic Graphs (Details)

\[
\begin{align*}
H_k &= M \cdot H_k \\
H_{k-1} &= V \left( I \otimes e_k e_k^T \right) V^{-1} \\
H_k(e^{j\omega}) &= \begin{cases} 
1, & \frac{2\pi k}{M} \leq \omega \leq \frac{2\pi (k+1)}{M}, \\
0, & \text{otherwise},
\end{cases}
\end{align*}
\]

\[
H_k = H_k(A) = h_k(0) A^0 + h_k(1) A^1 + \ldots + h_k(L) A^L
\]

*A needs to have distinct eigenvalues.*

---


Is $M$-Block Cyclic Property Necessary?

$M$-Block Cyclic \[\iff\]
\[
\begin{align*}
\text{Eigenvector Property} : & \quad v_{i,j+k} = \Omega^k v_{i,j} \\
\text{Eigenvalue Property} : & \quad \lambda_{i,j+k} = w^k \lambda_{i,j}
\end{align*}
\]

---

Is $M$-Block Cyclic Property Necessary?

$M$-Block Cyclic $\Leftrightarrow \begin{cases} 
\text{Eigenvector Property} : & v_{i,j+k} = \Omega^k v_{i,j} \\
\text{Eigenvalue Property} : & \lambda_{i,j+k} = w^k \lambda_{i,j}
\end{cases}$

---

Is $M$-Block Cyclic Property Necessary?

$M$-Block Cyclic $\iff$

\[
\begin{align*}
\text{Eigenvector Property} : & \quad v_{i,j+k} = \Omega^k v_{i,j} \\
\text{Eigenvalue Property} : & \quad \lambda_{i,j+k} = w^k \lambda_{i,j}
\end{align*}
\]

For any polynomial $H(A)$

\[H(Q AQ^{-1}) = Q H(A) Q^{-1}\] for any invertible $Q$

---


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Is $M$-Block Cyclic Property Necessary?

$M$-Block Cyclic \[\leftrightarrow\] \[\begin{align*}
&\text{Eigenvector Property:} \quad v_{i,j+k} = \Omega^k v_{i,j} \\
&\text{Eigenvalue Property:} \quad \lambda_{i,j+k} = w^k \lambda_{i,j}
\end{align*}\]

For any polynomial $H(A)$

$$H(QAQ^{-1}) = Q H(A) Q^{-1} \text{ for any invertible } Q$$

---

Is $M$-Block Cyclic Property Necessary?

$M$-Block Cyclic $\iff \begin{cases} \text{Eigenvector Property:} & v_{i,j+k} = \Omega^k v_{i,j} \\ \text{Eigenvalue Property:} & \lambda_{i,j+k} = w^k \lambda_{i,j} \end{cases}$

For any polynomial $H(A)$

$$H(QAQ^{-1}) = QH(A)Q^{-1} \text{ for any invertible } Q$$

$\tilde{D}$ and $\tilde{U}$ have higher complexity, but no restrictive assumptions on $A$.

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Outline

1. Introduction to Graph Signal Processing
   - Graph Signals

2. Purpose: $M$-Channel Filter Banks
   - Extending Classical Tools to Graph Case

3. Multirate Processing of Graph Signals
   - Decimation & Expansion
   - $M$-Block Cyclic Graphs
   - Concept of Spectrum Folding
   - More Results for $M$-Block Cyclic

4. Some Examples
Examples

(a) Signal

(b) Output of Channel-1

(c) Output of Channel-2

(d) Output of Channel-3

---


Examples

(i) Signal\(^8\) \(^9\)  
(j) Output of Channel-1


Some Examples

Examples

(q) Signal

(r) Output of Channel-1

(s) Output of Channel-2

(t) Output of Channel-3


Some Examples

Examples

(y) Signal

(z) Output of Channel-1

() Output of Channel-2

() Output of Channel-3

() Signal

() Output of Channel-1

() Output of Channel-2

() Output of Channel-3

---


Conclusions

- Brute-Force Filter Banks
- Decimator
- $M$-Block Cyclic Graph
  - Unique Eigenvalue-Eigenvector Structure
- Spectrum Folding
  - Decimation-then-Expansion
  - Bandlimited Signals
  - Interpolation
- $M$-Channel Filter Banks
- Further Directions
  - How does this compare with alternative ways?
  - From ideal to non-ideal?
Conclusions

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Any questions?
Please email me: oteke@caltech.edu