Are there approximate Fast Fourier Transform on graphs?

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**Objective**

**Goal:** Enable Fast Fourier Transform (FFT) and fast filtering on large graphs.

**Approach:** Provide a general method for approximating the graph Fourier matrix $U$, giving approximations $\hat{U}$ that can be applied rapidly.

**Graph Fourier transform**

Let $L \in \mathbb{R}^{n \times n}$ be the laplacian matrix of a graph, and $U \in \mathbb{R}^{n \times n}$ its eigenvectors matrix. Let $x \in \mathbb{R}^n$ be a signal on the graph, and $y \in \mathbb{R}^n$ be its Fourier transform, we have:

$$y = U^T x$$

$$x = Uy.$$  

The matrix $U$ being dense in general, the Fourier transform costs $O(n^2)$ arithmetic operations.

**Fast transforms**

Many widely used transforms (classical Fourier, wavelets, DCT, etc.) are paired with a fast algorithm, exploiting the factorizability of the associated matrix $A$ into sparse factors,

$$A = \prod_{j=1}^{J} S_j.$$  

This factorizability is necessary and sufficient for a fast linear algorithm to exist. In the case of the classical Fourier transform, $A$ can be factorized into $J = \log_2(n)$ factors, each having $2n$ nonzero entries.

**FA$\mu$ST approximations**

We approximate $U$ using Flexible Approximate MUlti-layer Sparse Transforms (FA$\mu$ST) [1]:

$$U \approx \hat{U} = \prod_{j=1}^{J} S_j,$$

allowing to compute approximate Fourier transformations ($\hat{U^T x}$ and $\hat{U}y$) in only $O(\sum_{j=1}^{J} \|S_j\|_0)$ arithmetic operations.

**Optimization problems**

We consider two optimization problems:

- Approximate factorization of $U$ (giving $\hat{U}_{\text{fact}}$):
  
  $$\min_{S_j \in S_j} \frac{1}{2} \|U - S_j \ldots S_1\|^2_F,$$

  subject to $S_j \in S_j$, $\forall j \in \{1, \ldots, J\}$.

- Approximate diagonalization of $L$ (giving $\hat{U}_{\text{diag}}$):
  
  $$\min_{S_j \in S_j, D \in D} \frac{1}{2} \|L - S_j \ldots S_1 D S_1^T \ldots S_j^T\|^2_F,$$

  subject to $S_j \in S_j$, $\forall j \in \{1, \ldots, J\}$.

both tackled with the hierarchical strategy of [1].

**Main Contribution**

A flexible approach that allows to get FA$\mu$STs with computational complexities $O(n^\alpha)$, $1 < \alpha < 2$, approximating well the Fourier transform of many classical families of graphs.

**Experimental validation**

**Filtering experiment**

Table 1: Filtering results, the SNRs in dBs and in average over 100 independently drawn signals for each noise level are given.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Noise level (dB)</th>
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<tbody>
<tr>
<td>0.3</td>
<td>1.82, -0.68, -2.65</td>
</tr>
<tr>
<td>0.4</td>
<td>5.11, 4.57, 3.89</td>
</tr>
<tr>
<td>0.5</td>
<td>4.04, 3.62, 3.11</td>
</tr>
</tbody>
</table>

**Future work**

- Designing a method that does not require a pre-computed diagonalization of the Laplacian $L$.
- Imposing orthogonal FA$\mu$STs, to ensure perfect reconstruction ($\hat{U}^T \hat{U} = I_d$).

**References**

[1] Luc Le Magoarou and Rémi Gribonval.

**Acknowledgments**

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