NON-MONOTONE QUADRATIC POTENTIAL GAMES

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NON CONVEX QUADRATIC PROBLEMS

- We analyze the fundamental theory of Quadratic Problems (QP) **with single constraints.**
- These problems possess *strong duality* under Slater's condition.
- Quadratic Problem:

$$\begin{aligned} \min_{x} \quad & x^T A_0 x + 2b_0 x + c_0 \\ \text{s.t.} \quad & x^T A_1 x + 2b_1 + c_1 \le 0 \end{aligned} \qquad (1)$$

  where $A_i \in \mathbf{S}^n$, $b_i \in \mathbf{R}^n$, $c_i \in \mathbf{R}$.
- Lagrangian: $L(x,\lambda) = x^T(A_0 + \lambda A_1)x + 2(b_0 + \lambda_0 + \lambda b_1)^T x + c_0 + \lambda c_1$
- Dual problem:

$$\begin{aligned} \max_{\lambda,\gamma} \quad & \gamma \\ \text{s.t.} \quad & \lambda \ge 0 \\ & \begin{bmatrix} A_0 + \lambda A_1 & b_0 + \lambda b_1 \\ (b_0 + \lambda b_1)^T & c_0 + \lambda c_1 - \gamma \end{bmatrix} \succeq 0 \end{aligned} \qquad (2)$$

- *Strong duality*: the optimal values of both problems coincide.

POTENTIAL GAME

Consider a strategic non-cooperative game $\mathcal{G} = \{\mathcal{Q}, \mathcal{X}, \{f_i\}_{i\in\mathcal{Q}}\}$ where

- $\mathcal{Q}$ is the set of Q players.
- $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_Q \subset \mathbb{R}^n$ is the set of pure strategies, i.e. $x = (x_i)_{i\in\mathcal{Q}} \in \mathcal{Q}$.
- function $f_i : \mathcal{X}_i \to \mathbb{R}$ is the payoff for player i.
- A game $\mathcal{G}$ is called an exact potential game if there exists a function $V(x)$ such that

$$f_i(x_i, x_{-i}) - f_i(y_i, x_{-i}) = V(x_i, x_{-i}) - V(y_i, x_{-i}) \qquad \forall x_i \in \mathcal{X}_i, x_{-i} \in \mathcal{X}_{-i}. \qquad (4)$$

- EQUIVALENT QUADRATIC POTENTIAL PROBLEM:

$$\begin{aligned} \min_{x} \quad & V(x) = x^T A_0 x + 2b_0^T x + \mathbf{1}_{n\times 1}^T c_0 \\ \text{s.t.} \quad & x_i^T A_1^i x_i + 2b_{1i}^T x_i + c_{1i} \le 0 \qquad \forall i \in \mathcal{Q}. \end{aligned} \qquad (5)$$

- **Potential problem has multiple constraints.**
- Solving the potential problem (5) provides an NE solution of the game.
- Notation: $b_0 = (b_{0i})_{i=1}^Q$, $b_1 = (b_{1i})_{i=1}^Q$, $c_0 = (c_{0i})_{i=1}^Q$, $A_1 = \text{diag}[A_1^1, \ldots, A_1^i, \ldots, A_1^Q]$, $D(\boldsymbol{\lambda}) = \text{diag}[\boldsymbol{\lambda}] \otimes I_{n\times n}$, $c_1 = (c_{1i})_{i=1}^Q$, "diag" is the block diagonal matrix operator and "$\otimes$" is the Kronecker product.

ANALYSIS RESULTS OVER THE POTENTIAL PROBLEM

- **Strong duality**: primal problem can be solved through the dual

$$q(\boldsymbol{\lambda}) = \begin{cases} -(b_0 + D(\boldsymbol{\lambda})b_1)^T (A_0 + D(\boldsymbol{\lambda})A_1)^\dagger (b_0 + D(\boldsymbol{\lambda})b_1) \\ +\mathbf{1}_{n\times 1}^T c_0 + \boldsymbol{\lambda}^T c_1 & \text{if } A_0 + D(\boldsymbol{\lambda})A_1 \succeq 0 \\ & \text{and } (b_0 + D(\boldsymbol{\lambda})b_1) \in \mathcal{R}(A_0 + D(\boldsymbol{\lambda})A_1) \\ -\infty & \text{otherwise.} \end{cases}$$

- **Coercivity**: $\lim_{\|\boldsymbol{\lambda}\|\to\infty} q(\boldsymbol{\lambda}) \to -\infty$
- **Existence of solution** $\Leftrightarrow$ **existence of NE** $\Leftrightarrow$ $\{\boldsymbol{\lambda} \in \mathbb{R}_+^Q \mid A_0 + D(\boldsymbol{\lambda})A_1 \succeq 0\}$ is nonempty.

ALGORITHMS

- **Centralized**: solve concave problem $q(\boldsymbol{\lambda})$ and calculate

$$\mathbf{x}^* \in -(A_0 + D(\boldsymbol{\lambda}^*)A_1)^\dagger (b_0^T + b_1^T D(\boldsymbol{\lambda})) + \mathcal{N}(A_0 + D(\boldsymbol{\lambda}^*)A_1)$$

  where x^* is an NE of $\mathcal{G}_p$, and $\mathcal{N}(Z)$ represents the nullspace of Z.
- **Distributed**:

Algorithm 1 Distributed Jacobi scheme ($A_1^i \succ 0 \quad \forall i \in \mathcal{Q}$)

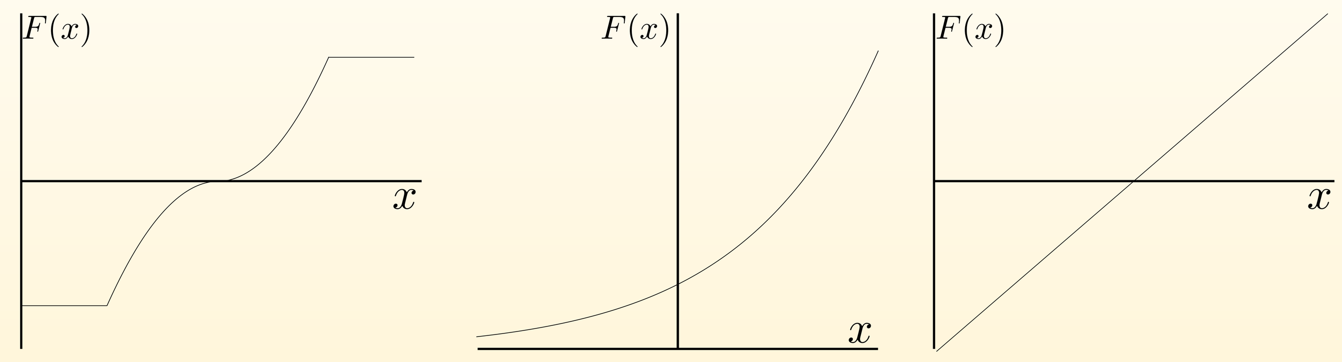
Initialize $(x_i^0)_i$. Determine $\lambda_i^{\min}$ $\forall i$. Set $k \leftarrow 0$.
while $\|x^k - x^{k-1}\| \ge \varepsilon_{\text{outer}}$ **do**
    Set $k \leftarrow k+1$.
    Calculate $b_{gi} = b_{0i} + \sum_{j\neq i} A_0^{ij} x_j, \forall i$ *//Mix strategies*
    for $i \in \mathcal{Q}$ **do**
        Set $\underline{\lambda}_i = \lambda_i^{\min}$, $\overline{\lambda}_i = 2\lambda_i^{\min} + 1$, and $\overline{x}_i = \hat{x}_i(\overline{\lambda}_i, b_{gi})$.
        while $h_i(\overline{x}_i) \ge 0$ **do** *//Find bisection limits*
            Update $\underline{\lambda}_i = \overline{\lambda}_i$; $\overline{\lambda}_i = 2\underline{\lambda}_i$. Solve $\overline{x}_i = \hat{x}_i(\overline{\lambda}_i, b_{gi})$
        Set $\Psi_{\text{cost}} \ge \epsilon_{\text{inner}}$
        while $|\Psi_{\text{cost}}| \ge \varepsilon_{\text{inner}}$ **do** *//Perform bisection steps*
            Set $\lambda_i^k = \frac{1}{2}(\overline{\lambda}_i + \underline{\lambda}_i)$, determine $x_i^k = \hat{x}_i(\lambda_i^k, b_{gi})$.
            if $h_i(\overline{x}_i) \le 0$, **then** $\overline{\lambda}_i = \lambda_i$
            else, $\underline{\lambda}_i = \lambda_i$.
            if $\lambda_i^k > 0$, **then** $\Psi_{\text{cost}} = h_i(x_i^k)$ *//Slackness violation*
            else, $\Psi_{\text{cost}} = 0$ *//case $\lambda_i \approx 0$*
    Solve $(\lambda_i^k)_{i=1}^Q = \Pi_\Gamma((\lambda_i^k)_{i=1}^Q)$, update $x_i^k = \hat{x}_i(\lambda_i^k, b_{gi})$.

MONOTONICITY IN GAMES

- Given a convex subspace $\mathcal{X} \subseteq \mathbb{R}^n$, a mapping $\mathbf{F} : \mathcal{X} \to \mathbb{R}^\mathbf{n}$ is monotone if:

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^\mathbf{T}(\mathbf{x} - \mathbf{y}) \ge \mathbf{0}, \qquad \forall \mathbf{x}, \mathbf{y} \in \mathcal{X} \qquad (3)$$

- Examples (borrowed from [4]):



- Monotone games $\Rightarrow$ existence of NE & algorithms that converge to NE.
- Coupling among players is limited.
- **Monotonicity is a strong requirement.**

PROBLEM FORMULATION

Given a set of players $\mathcal{Q} = \{1, \ldots, Q\}$, we introduce the quadratic potential game $\mathcal{G}_p$ where every player $i \in \mathcal{Q}$ has to solve

$$\forall i \in \mathcal{Q} \quad \begin{cases} \min_{x_i \in \mathbb{R}^n} & f_i(x_i, x_{-i}) = x_i^T A_0^{ii} x_i + 2\sum_{j\neq i} x_j^T A_0^{ij} x_i + 2b_{0i}^T x_i + c_{0i} \\ \text{s.t.} & h_i(x_i) = x_i^T A_1^i x_i + 2b_{1i}^T x_i + c_{1i} \le 0 \end{cases} \qquad (6)$$

- A_0^{ii}, $A_1^i \in \mathcal{S}^n$, $A_0^{ij} \in \mathbb{R}^{n\times n}$, $\mathcal{S}^n$ is the set of symmetric matrices of size n;
- b_{0i}, $b_{1i} \in \mathbb{R}^n$ are column vectors;
- c_{0i}, $c_{1i} \in \mathbb{R}$ are scalar numbers.
- The game is potential if, and only if, its Jacobian given by

$$A_0 = \begin{bmatrix} A_0^{11} & \cdots & A_0^{1N} \\ \vdots & \ddots & \vdots \\ A_0^{N1} & \cdots & A_0^{NN} \end{bmatrix},$$

  is symmetric, i.e., $A_0^{ij} = (A_0^{ji})^T$.

A_0^{ii}, A_1^i do not need to be positive semidefinite. The problems do not need to be convex.

The quadratic game does not need to be monotone.

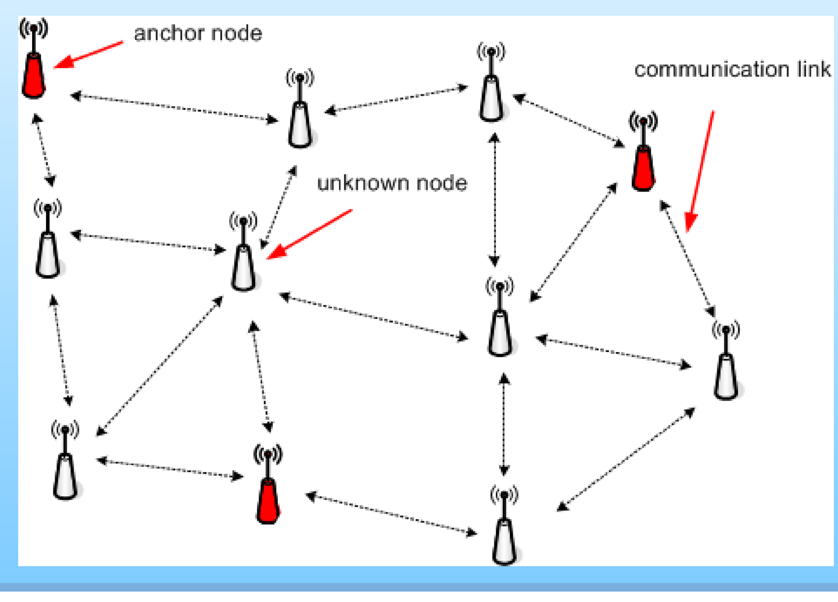
APPLICATIONS

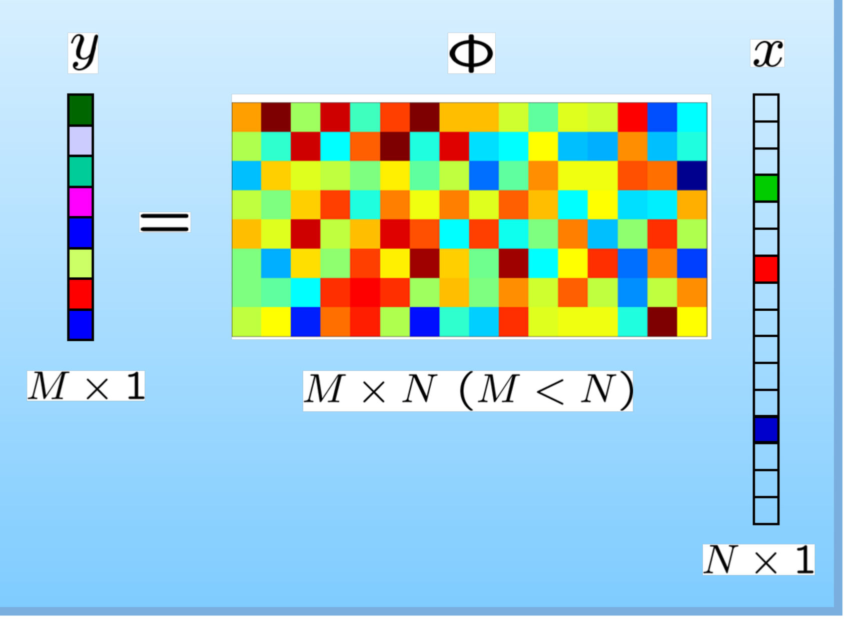
- Optimal localization (Non-Convex)

$$\min_{x\in\mathbb{R}^n} \sum_{i\in\mathcal{Q}} (d_i^2 - \|x - y_i\|^2)^2$$

- Robust Least Squares (MinMax)

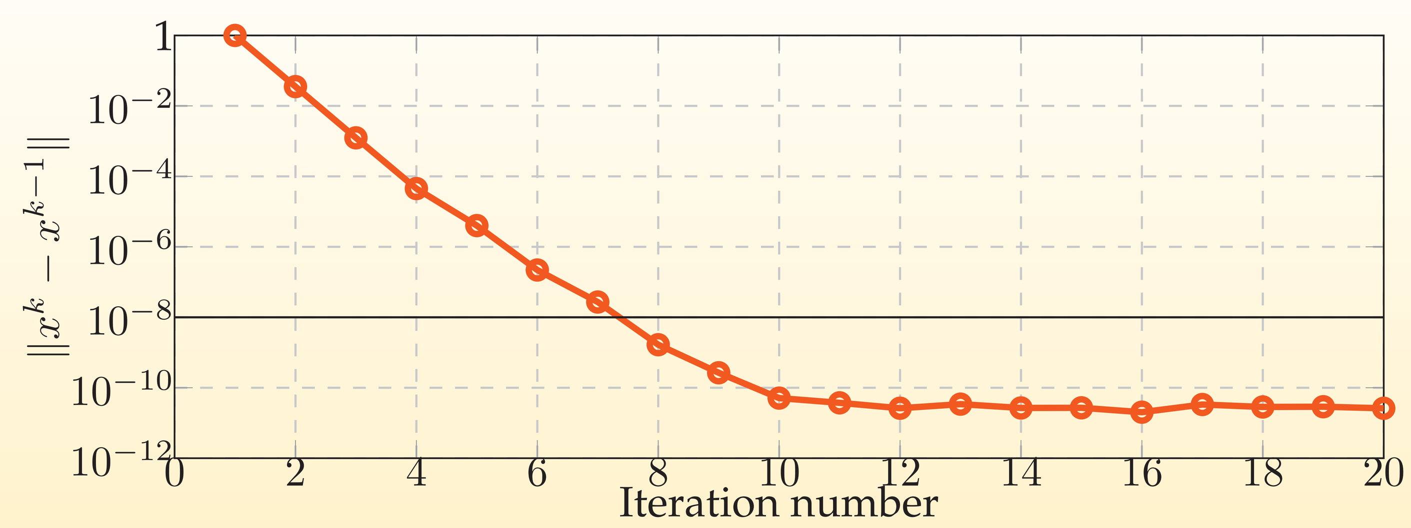
$$\min_{x\in\mathbb{R}^n} \max_{\{\|(\Delta_i, \delta_i)\| \le \Gamma_i\}_{i\in\mathcal{Q}}} \|(A + \Delta)x - \delta - b\|$$





SIMULATIONS

- 200 simulated games, $Q = 10$ payers, and $n = 4$.



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