Anti-sparse Representation for Continuous Function by Dual Atomic Norm with Application in OFDM

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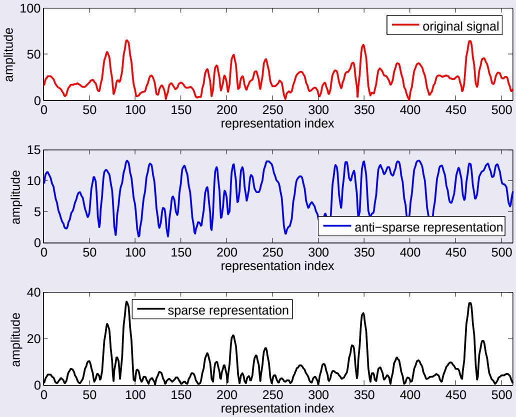
Contents

Background

Anti-sparse v.s. sparse representation



Background

Anti-sparse representation

- Energy evenly allocated on the entire domain
- Amplitude under control
- Applied in communucation systems and control systems
- Vector anti-sparse representation, also known as spread/democratic representation [Fuchs; Studer, Goldstein, Yin, Baraniuk]

$$\min_{\mathbf{x}\in\mathbb{C}^{\mathbf{N}}} \left\{ \|\mathbf{x}\|_\infty := \max_{i=1,\ldots,N} |\mathbf{x}_i| : \quad \|\mathbf{y} - D\mathbf{x}\|_2 \leq \varepsilon_1 \right\} \tag{1}$$

in which $D \in \mathbb{C}^{M\times N}$ is a redundant dictionary $(M < N)$

Background

Atomic norm (Minkowski functional)

- Atomic norm of a vector $\mathbf{x} \in \mathbb{C}^{\mathbf{N}}$

$$\|\mathbf{x}\|_{\mathcal{A}} := \inf_{\alpha \geq 0} \left\{ \mathbf{x} \in \alpha \cdot \mathrm{conv}(\mathcal{A}) \right\}$$

  - $\mathcal{A} \subset \mathbb{C}^N$: bounded symmetric set
  - $\mathrm{conv}(\mathcal{A})$: convex hull of $\mathcal{A}$
- Dual atomic norm of $\mathbf{x} \in \mathbb{C}^{\mathbf{N}}$

$$\|\mathbf{x}\|_{\mathcal{A}}^{*} := \sup_{\mathbf{a} \in \mathcal{A}} \langle \mathbf{a}, \mathbf{x} \rangle \tag{2}$$

Formulation

primal problem

$$\min_{\mathbf{f}\in\mathbb{C}^N} \left\{ \|\mathbf{f}\|_{\mathcal{A}}^* : \ \ \|(\mathbf{f}-\mathbf{f}_0)_U\|_2 \le \varepsilon_1, \|(\mathbf{f}-\mathbf{f}_0)_{U^C}\|_2 \le \varepsilon_2 \right\} \qquad (3)$$

- index set $U \subset \{1, 2, \cdots, N\}$, $|U| = M \le N$
- two deviation levels $\varepsilon_1 \le \varepsilon_2$
- $\mathcal{A} := \{\mathbf{a}(\omega) \in \mathbb{C}^N : \ \omega \in \Omega\}$
- $h(\omega) := \langle \mathbf{a}(\omega), \mathbf{f} \rangle, \omega \in \Omega$
- (3) equivalent to minimization of the infinite norm of $h(\omega)$

$$\min_{\mathbf{f}\in\mathbb{C}^N} \{ \|h(\omega)\|_\infty : \ h(\omega) = \langle \mathbf{a}(\omega), \mathbf{f} \rangle, \qquad (4)$$
$$\|(\mathbf{f}-\mathbf{f}_0)_U\|_2 \le \varepsilon_1, \|(\mathbf{f}-\mathbf{f}_0)_{U^C}\|_2 \le \varepsilon_2 \}$$

Dual problem and bounds on solution

Lemma

The dual problem of problem (3) *is*

$$-\min_{\mathbf{z}\in\mathbb{C}^{\mathbf{N}}}\left\{\varepsilon_1\|\mathbf{z}_U\|_2+\varepsilon_2\|\mathbf{z}_{U^C}\|_2-\mathcal{R}(\mathbf{z}^*\mathbf{f}_0):\quad\|\mathbf{z}\|_{\mathcal{A}}\leq 1\right\} \tag{5}$$

- convex hull can have semi-definite characterization

Dual problem and bounds on solution

Proposition

If $\|\mathbf{f}_0\|_2 \geq \varepsilon_2 \geq \varepsilon_1$, then the solution to the primal problem (3) $\hat{\mathbf{f}}$ satisfies that

$$\frac{\|\mathbf{f}_0\|_2 - \varepsilon_1 - (\varepsilon_2 - \varepsilon_1)\frac{\|(\mathbf{f}_0)_{U^C}\|_2}{\|\mathbf{f}_0\|_2}}{M_A} \leq \|\hat{\mathbf{f}}\|_{\mathcal{A}}^* \leq \frac{\|\mathbf{f}_0\|_2 - \varepsilon_1}{m_A},$$

in which M_A and m_A are the smallest and the largest real positive number such that $\forall \mathbf{v} \in \mathbb{C}^N$,

$$m_A\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_{\mathcal{A}} \leq M_A\|\mathbf{v}\|_2.$$

- invariant under shrinkage of $\mathcal{A}$
- smaller gap between the two sides, if the atomic norm ball is more isotropic

Dual atomic norm minimization for OFDM PAPR reduction

Define an OFDM signal $s(t)$ with f_n on the n-th sub-channel

$$s(t) := \sum_{n=0}^{N-1} f_n \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T}tn}$$

- $\mathcal{A} = \{\mathbf{a}(t,\phi) = \mathrm{e}^{\mathrm{j}\phi}[1, \mathrm{e}^{-\mathrm{j}\frac{t}{2\pi T}}, \cdots, \mathrm{e}^{-\mathrm{j}\frac{t(N-1)}{2\pi T}}]^{\mathrm{T}}, \phi \in [0, 2\pi), t \in [0, T)\}$
- $h(t,\phi) = \mathcal{R}\left(\mathrm{e}^{-\mathrm{j}\phi} \sum_{n=0}^{N-1} f_n \mathrm{e}^{-\mathrm{j}\frac{nt}{2\pi T}}\right), \quad \phi \in [0, 2\pi), t \in [0, T)$
- $\sup_{\phi \in [0,2\pi)} h(t,\phi) = \left|\sum_{n=0}^{N-1} f_n \mathrm{e}^{-\mathrm{j}\frac{2\pi t n}{T}}\right| = |s(t)|$
- $\|\mathbf{f}\|_{\mathcal{A}}^{*} = \sup_{t,\phi} h(t,\phi) = \|s(t)\|_{\infty}$
- tone reservation (not compulsory)
  - U: the unreserved tones, $(\mathbf{f}_0)_{U^C} = \mathbf{0}$
  - ε_1 small enough to control the error symbol rate
  - ε_2 can be much larger

Dual atomic norm minimization for OFDM PAPR reduction

- solution $\hat{\mathbf{f}}$ gives $\hat{s}(t) = \sum_{n=0}^{N-1} \hat{f}_n \mathrm{e}^{-\mathrm{j}t\frac{2\pi n}{T}}$
- $\hat{f}_n$ no longer a symbol, but $|\hat{f}_n - f_{0n}|$ for $n \in U$ should be smaller than the quantization threshold

Corollary

If $\|\mathbf{f}_0\|_2 \geq \varepsilon_2 \geq \varepsilon_1$ *and* $(\mathbf{f}_0)_{U^C} = \mathbf{0}$*, then the peak-to-average ratio of the continuous function corresponding to the solution to problem* (3) $\hat{h}(\omega, \phi) = \langle a(\omega, \phi), \hat{\mathbf{f}} \rangle$ *satisfies that*

$$\frac{\|\hat{h}(\omega)\|_\infty}{\|\hat{h}(\omega)\|_2} = \frac{\|\hat{\mathbf{f}}\|_{\mathcal{A}}^*}{\sqrt{N}\pi\|\hat{\mathbf{f}}\|_2} \leq \frac{\|\mathbf{f}_0\|_2 - \varepsilon_1}{\pi\|\hat{\mathbf{f}}\|_2} \leq \frac{1}{\pi}.$$

Solving method

- atomic norm as semi-definite problem

$$\|\mathbf{z}\|_{\mathcal{A}} = \inf_{\mathbf{u}\in\mathbb{C}^{\mathbf{N}},\alpha\in\mathbb{R}} \left\{ \frac{1}{2N}\mathrm{tr}(\mathrm{Toep}(\mathbf{u})) + \frac{\alpha}{2} : \begin{bmatrix} \mathrm{Toep}(\mathbf{u}) & \mathbf{z} \\ \mathbf{z}^* & \alpha \end{bmatrix} \succeq 0 \right\}$$

  - $\mathrm{Toep}(\mathbf{u})$: the symmetric Toeplitz matrix generated by $\mathbf{u}$
- due to the closeness of the constraint set,

$$\inf_{\mathbf{u},\alpha} \left\{ \frac{1}{2N}\mathrm{tr}(\mathrm{Toep}(\mathbf{u})) + \frac{\alpha}{2} : \begin{bmatrix} \mathrm{Toep}(\mathbf{u}) & \mathbf{z} \\ \mathbf{z}^* & \alpha \end{bmatrix} \succeq 0 \right\} \leq 1$$

is equivalent to $\exists \mathbf{u}, \alpha$ such that

$$\frac{1}{2N}\mathrm{tr}(\mathrm{Toep}(\mathbf{u})) + \frac{\alpha}{2} \leq 1$$
$$\begin{bmatrix} \mathrm{Toep}(\mathbf{u}) & \mathbf{z} \\ \mathbf{z}^* & \alpha \end{bmatrix} \succeq 0.$$

Solving method

- dual problem (5) transformed to

$$
-\min_{\mathbf{z},\mathbf{u}\in\mathbb{C}^N,\alpha\in\mathbb{R}} \varepsilon_1\|\mathbf{z}_U\|_2+\varepsilon_2\|\mathbf{z}_{U^C}\|_2-\mathcal{R}(\mathbf{z}^*\mathbf{f}_0) \tag{6}
$$

$$
\text{s.t.}\quad \frac{1}{2N}\mathrm{tr}(\mathrm{Toep}(\mathbf{u}))+\frac{\alpha}{2}\le 1,\ \begin{bmatrix}\mathrm{Toep}(\mathbf{u}) & \mathbf{z}\\ \mathbf{z}^* & \alpha\end{bmatrix}\succeq 0,
$$

  which can be solved by SDP
- strong duality gives the primal solution

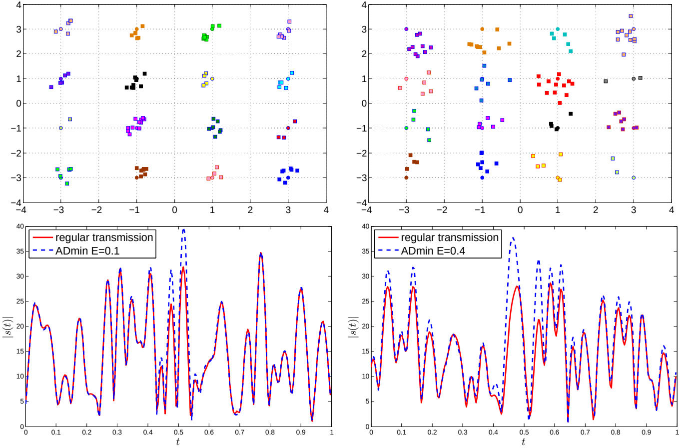
$$
\hat{\mathbf{f}}_U=(\mathbf{f}_0)_U-\varepsilon_1\frac{\hat{\mathbf{z}}_U}{\|\hat{\mathbf{z}}_U\|_2},\ \hat{\mathbf{f}}_{U^C}=-\varepsilon_2\frac{\hat{\mathbf{z}}_{U^C}}{\|\hat{\mathbf{z}}_{U^C}\|_2}. \tag{7}
$$

General settings

- $s(t)$: 16 QAM OFDM signal
- $[\mathbf{f}_0]_n, n \in U$: uniformly randomly chosen from symbol set
- PAPR: calculated by 4 times over-sampling the transmission signal
- error symbol rate: after assigning each entry of $\hat{\mathbf{f}}$ to the nearest constellation point
- reserved tones uniformly randomly chosen
- $\varepsilon_1 = \sqrt{EM}$, $\varepsilon_2 = \sqrt{100(N-M)}$
- dual problem (6) solved by SDP tool box in CVX [Grant, Boyd]

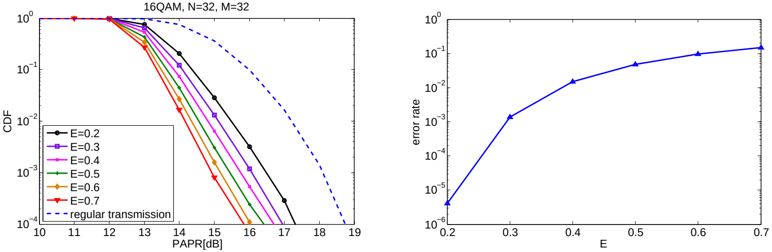
Atomic dual norm minimization

$M = N = 32$, ε_1 v.s. PAPR reduced results



- first row: $E = 0.1$, second row: $E = 0.4$
- left figures: constellations from 3 random trials
- right figures: amplitudes of the transmission signals

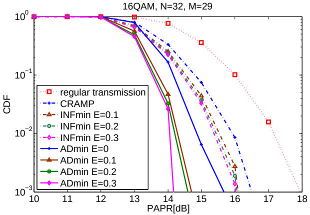
Atomic dual norm minimization

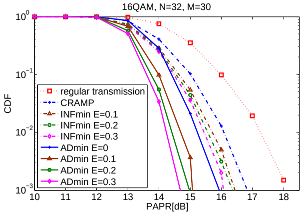


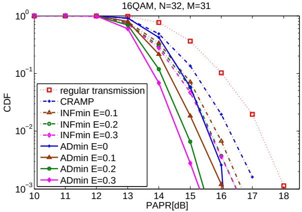
- 10^5 trials for each E from 0.2 to 0.7 with increment 0.1
- left figure: cumulation density function of PAPR
- right figure: cumulation density function of error symbol rate

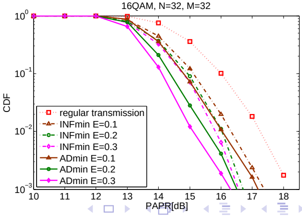
Comparison with vector ℓ_∞ norm minimization

- cumulation density function of PAPR for 10^4 trials
- vector ℓ_∞: CRAMP [Studer, Goldstein, Yin, Baraniuk] $(\varepsilon_1 = 0)$ and INFmin $(\varepsilon_1 \neq 0)$
- atomic dual norm: ADmin
- $N = 32,\ M = 29, 30, 31, 32$









Comparison with vector ℓ_∞ norm minimization

Table : error symbol rate for 10^4 trials. $N = 32$.

		E=0.3	E=0.2	E=0.1	E=0
M=29	ADmin	0.0015	0	0	0
	INFmin	0.0029	7.8125e-06	0	-
	CRAMP	-	-	-	0
M=30	ADmin	0.0016	9.3750e-06	0	0
	INFmin	0.0024	1.9531e-05	0	-
	CRAMP	-	-	-	0
M=31	ADmin	0.0016	0	0	0
	INFmin	0.0030	1.1719e-05	0	-
	CRAMP	-	-	-	0
M=32	ADmin	0.0015	3.9063e-06	0	-
	INFmin	0.0021	3.9063e-06	0	-

Conclusion

- Anti-sparse representation for a class of continuous functions
- Dual atomic norm minimization problem
- Dual problem and bounds on solution
- Application in OFDM PAPR: atom set composed of complex exponentials, dual problem solved by SDP
- Experiments in 16 QAM OFDM PAPR reduction: shows advantages in both PAPR and error rate than the vector ℓ_∞ method

Thanks

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